

# HARTSHORNE'S QUESTIONS ON COFINITENESS AND CORRELATION BETWEEN THEM

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**ABSTRACT.** Let  $\mathfrak{a}$  be an ideal of a commutative noetherian ring  $R$ . In the case  $\text{cd}(\mathfrak{a}, R) \leq 1$ , we show that the subcategory of  $\mathfrak{a}$ -cofinite  $R$ -modules is abelian. Using this and the technique of way-out functors, we show that if  $\text{cd}(\mathfrak{a}, R) \leq 1$ , or  $\dim(R/\mathfrak{a}) \leq 1$ , or  $\dim(R) \leq 2$ , then the local cohomology module  $H_{\mathfrak{a}}^i(X)$  is  $\mathfrak{a}$ -cofinite for every  $R$ -complex  $X$  with finitely generated homology modules and every  $i \in \mathbb{Z}$ . Deploying this, we answer Hartshorne's third question in the three aforementioned cases. Further, we reveal a connection between Hartshorne's questions on cofiniteness.

## 1. Introduction

Throughout this paper,  $R$  denotes a commutative noetherian ring with identity and  $\mathcal{M}(R)$  flags the category of  $R$ -modules.

In 1969, Hartshorne introduced the notion of cofiniteness for modules and complexes; see [Ha1]. He defined an  $R$ -module  $M$  to be  $\mathfrak{a}$ -cofinite if  $\text{Supp}_R(M) \subseteq V(\mathfrak{a})$  and  $\text{Ext}_R^i(R/\mathfrak{a}, M)$  is finitely generated for every  $i \geq 0$ . Moreover, in the case where  $R$  is an  $\mathfrak{a}$ -adically complete regular ring of finite Krull dimension, he defined an  $R$ -complex  $X$  to be  $\mathfrak{a}$ -cofinite if  $X \simeq \mathbf{R}\text{Hom}_R(Y, \mathbf{R}\Gamma_{\mathfrak{a}}(R))$  for some  $R$ -complex  $Y$  with finitely generated homology modules. He then proceeded to pose three questions in this direction which we paraphrase as follows.

**Question 1.1.** *Is the local cohomology module  $H_{\mathfrak{a}}^i(M)$ ,  $\mathfrak{a}$ -cofinite for every finitely generated  $R$ -module  $M$  and every  $i \geq 0$ ?*

**Question 1.2.** *Is the category  $\mathcal{M}(R, \mathfrak{a})_{\text{cof}}$  consisting of  $\mathfrak{a}$ -cofinite  $R$ -modules an abelian subcategory of  $\mathcal{M}(R)$ ?*

**Question 1.3.** *Is it true that an  $R$ -complex  $X$  is  $\mathfrak{a}$ -cofinite if and only the homology module  $H_i(X)$  is  $\mathfrak{a}$ -cofinite for every  $i \in \mathbb{Z}$ ?*

By providing a counterexample, Hartshorne showed that the answers to these questions are negative in general; see [Ha1, Section 3]. However, he established affirmative answers to these questions in the case where  $\mathfrak{a}$  is a principal ideal generated by a nonzerodivisor and  $R$  is an  $\mathfrak{a}$ -adically complete regular ring of finite Krull dimension, and also in the case where  $\mathfrak{a}$  is a prime ideal with  $\dim(R/\mathfrak{a}) = 1$  and  $R$  is a complete regular local ring; see [Ha1, Propositions 6.1 and 6.2, Corollary 6.3, Theorem 7.5, Proposition 7.6 and Corollary 7.7].

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Since then many papers are devoted to study his first two questions; see for example [HK], [DM], [Ka1], [Ka2], [Me1], [Me2]. These results were extended in several stages to take the following form:

**Theorem 1.4.** *Let  $\mathfrak{a}$  be an ideal of  $R$  such that either  $\text{ara}(\mathfrak{a}) \leq 1$ , or  $\dim(R/\mathfrak{a}) \leq 1$ , or  $\dim(R) \leq 2$ . Then  $H_{\mathfrak{a}}^i(M)$  is  $\mathfrak{a}$ -cofinite for every finitely generated  $R$ -module  $M$  and every  $i \geq 0$ , and  $\mathcal{M}(R, \mathfrak{a})_{\text{cof}}$  is an abelian subcategory of  $\mathcal{M}(R)$ .*

For the case  $\text{ara}(\mathfrak{a}) \leq 1$ , refer to [Ka2, Theorem 1] and [Ka1, Theorem 2.1]. For the case  $\dim(R/\mathfrak{a}) \leq 1$ , see [Me1, Theorem 2.6 and Corollary 2.12], [BNS, Corollary 2.8], and [BN, Corollary 2.7]. For the case  $\dim(R) \leq 2$ , observe [Me2, Theorem 7.10] and [Me2, Theorem 7.4].

The significance of cofiniteness of the local cohomology modules mainly stems from the fact that if an  $R$ -module  $M$  is  $\mathfrak{a}$ -cofinite, then its set of associated primes is finite as well as all its Bass numbers and Betti numbers with respect to every prime ideal of  $R$ . It is worth mentioning that the investigation of such finiteness properties is a long-sought problem in commutative and homological algebra; see e.g. [HS] and [Ly].

In this paper, we deal with the above three questions. Theorems 2.2, 3.3, 3.5 and 4.3 are our main results.

In [PAB, Question 1], the authors asked: Is  $\mathcal{M}(R, \mathfrak{a})_{\text{cof}}$  an abelian subcategory of  $\mathcal{M}(R)$  for every ideal  $\mathfrak{a}$  of  $R$  with  $\text{cd}(\mathfrak{a}, R) \leq 1$ ? We answer this question affirmatively by deploying the theory of local homology; see Theorem 2.2. Note that there exists an inequality  $\text{cd}(\mathfrak{a}, R) \leq \text{ara}(\mathfrak{a})$  that can be strict; see Example 2.3.

It turns out that to establish the cofiniteness of  $H_{\mathfrak{a}}^i(X)$  for any  $R$ -complex  $X$  with finitely generated homology modules, all we need to know is the cofiniteness of  $H_{\mathfrak{a}}^i(M)$  for any finitely generated  $R$ -module  $M$  and the abelianness of  $\mathcal{M}(R, \mathfrak{a})_{\text{cof}}$ ; see Theorem 3.3. The crucial step to achieve this is to recruit the technique of way-out functors.

To be consistent in both module and complex cases, we define an  $R$ -complex  $X$  to be  $\mathfrak{a}$ -cofinite if  $\text{Supp}_R(X) \subseteq V(\mathfrak{a})$  and  $\mathbf{R}\text{Hom}_R(R/\mathfrak{a}, X)$  has finitely generated homology modules. Corollary 4.2 indicates that, for homologically bounded  $R$ -complexes, this definition coincides with that of Hartshorne. Questions 1.1 and 1.2 have been high-profile among researchers, whereas not much attention has been brought to Question 1.3. The most striking result on this question is [EK, Theorem 1] which confines itself to complete Gorenstein local domains and the case  $\dim(R/\mathfrak{a}) = 1$ . We answer Hartshorne's third question in the cases  $\text{cd}(\mathfrak{a}, R) \leq 1$ ,  $\dim(R/\mathfrak{a}) \leq 1$ , and  $\dim(R) \leq 2$  with no extra assumptions on  $R$ ; see Corollary 3.6 (ii). Having the results thus far obtained at our disposal, we show that the answers to Questions 1.1 and 1.2 are affirmative if and only if the answer to Question 1.3 is affirmative for all homologically bounded  $R$ -complexes; see Theorem 4.3.

## 2. Question 1.2

We need to work in the framework of the derived category  $\mathcal{D}(R)$ . For more information, refer to [AF], [Ha2], [Fo], [Li], and [Sp].

We let  $\mathcal{D}_{\square}(R)$  (res.  $\mathcal{D}_{\square}(R)$ ) denote the full subcategory of  $\mathcal{D}(R)$  consisting of  $R$ -complexes  $X$  with  $H_i(X) = 0$  for  $i \gg 0$  (res.  $i \ll 0$ ), and let  $\mathcal{D}_{\square}(R) := \mathcal{D}_{\square}(R) \cap \mathcal{D}_{\square}(R)$ . We further

let  $\mathcal{D}^f(R)$  denote the full subcategory of  $\mathcal{D}(R)$  consisting of  $R$ -complexes  $X$  with finitely generated homology modules. We also feel free to use any combination of the subscripts and the superscript as in  $\mathcal{D}_{\square}^f(R)$ , with the obvious meaning of the intersection of the two subcategories involved.

**Lemma 2.1.** *Let  $\mathfrak{a}$  be an ideal of  $R$  and  $X \in D_{\square}(R)$ . Then the following conditions are equivalent:*

- (i)  $\mathbf{R}\mathrm{Hom}_R(R/\mathfrak{a}, X) \in D^f(R)$ .
- (ii)  $\mathbf{L}\Lambda^{\mathfrak{a}}(X) \in \mathcal{D}_{\square}^f(\widehat{R}^{\mathfrak{a}})$ .

**Proof.** See [WW, Propositions 7.4]. □

In this section, we show that given an ideal  $\mathfrak{a}$  of  $R$  with  $\mathrm{cd}(\mathfrak{a}, R) \leq 1$ , the subcategory  $\mathcal{M}(R, \mathfrak{a})_{\mathrm{cof}}$  of  $\mathcal{M}(R)$  is abelian. This fact is proved in [PAB, Theorem 2.4], under the extra assumption that  $R$  is local. Here we relax this assumption. The tool here is the local homology functors.

Recall that the local homology functors are the left derived functors of the completion functor. More precisely,  $H_i^{\mathfrak{a}}(-) := L_i(\Lambda^{\mathfrak{a}}(-))$  for every  $i \geq 0$ , where  $\Lambda^{\mathfrak{a}}(M) := \widehat{M}^{\mathfrak{a}} = \varprojlim_n (M/\mathfrak{a}^n M)$  for any  $R$ -module  $M$ . Further, we remind the cohomological dimension of  $M$  with respect to  $\mathfrak{a}$  as

$$\mathrm{cd}(\mathfrak{a}, M) := \sup \left\{ i \in \mathbb{Z} \mid H_{\mathfrak{a}}^i(M) \neq 0 \right\}.$$

**Theorem 2.2.** *Let  $\mathfrak{a}$  be an ideal of  $R$ . Then the following assertions hold:*

- (i) *An  $R$ -module  $M$  with  $\mathrm{Supp}_R(M) \subseteq V(\mathfrak{a})$  is  $\mathfrak{a}$ -cofinite if and only if  $H_i^{\mathfrak{a}}(M)$  is a finitely generated  $\widehat{R}^{\mathfrak{a}}$ -module for every  $0 \leq i \leq \mathrm{cd}(\mathfrak{a}, R)$ .*
- (ii) *If  $\mathrm{cd}(\mathfrak{a}, R) \leq 1$ , then  $\mathcal{M}(R, \mathfrak{a})_{\mathrm{cof}}$  is an abelian subcategory of  $\mathcal{M}(R)$ .*

**Proof.** (i): By [GM, Theorem 2.5 and Corollary 3.2],  $H_i^{\mathfrak{a}}(M) = 0$  for every  $i > \mathrm{cd}(\mathfrak{a}, R)$ . Therefore, the assertion follows from Lemma 2.1.

(ii): Let  $M$  and  $N$  be two  $\mathfrak{a}$ -cofinite  $R$ -modules and  $f : M \rightarrow N$  an  $R$ -homomorphism. The short exact sequence

$$(2.2.1) \quad 0 \rightarrow \ker f \rightarrow M \rightarrow \mathrm{im} f \rightarrow 0,$$

gives the exact sequence

$$H_0^{\mathfrak{a}}(M) \rightarrow H_0^{\mathfrak{a}}(\mathrm{im} f) \rightarrow 0,$$

which in turn implies that  $H_0^{\mathfrak{a}}(\mathrm{im} f)$  is finitely generated  $\widehat{R}^{\mathfrak{a}}$ -module since  $H_0^{\mathfrak{a}}(M)$  is so. The short exact sequence

$$(2.2.2) \quad 0 \rightarrow \mathrm{im} f \rightarrow N \rightarrow \mathrm{coker} f \rightarrow 0,$$

gives the exact sequence

$$(2.2.3) \quad H_1^{\mathfrak{a}}(N) \rightarrow H_1^{\mathfrak{a}}(\mathrm{coker} f) \rightarrow H_0^{\mathfrak{a}}(\mathrm{im} f) \rightarrow H_0^{\mathfrak{a}}(N) \rightarrow H_0^{\mathfrak{a}}(\mathrm{coker} f) \rightarrow 0.$$

As  $H_0^{\mathfrak{a}}(N)$ ,  $H_0^{\mathfrak{a}}(\mathrm{im} f)$ , and  $H_1^{\mathfrak{a}}(N)$  are finitely generated  $\widehat{R}^{\mathfrak{a}}$ -modules, the exact sequence (2.2.3) shows that  $H_0^{\mathfrak{a}}(\mathrm{coker} f)$  and  $H_1^{\mathfrak{a}}(\mathrm{coker} f)$  are finitely generated  $\widehat{R}^{\mathfrak{a}}$ -modules, and thus  $\mathrm{coker} f$  is  $\mathfrak{a}$ -cofinite by (i). From the short exact sequence (2.2.2), we conclude that

$\text{im } f$  is  $\mathfrak{a}$ -cofinite, and from the short exact sequence (2.2.1), we infer that  $\ker f$  is  $\mathfrak{a}$ -cofinite. It follows that  $\mathcal{M}(R, \mathfrak{a})_{\text{cof}}$  is an abelian subcategory of  $\mathcal{M}(R)$ .  $\square$

It is well-known that  $\text{cd}(\mathfrak{a}, R) \leq \text{ara}(\mathfrak{a})$ . On the other hand, the following example shows that an ideal  $\mathfrak{a}$  of  $R$  with  $\text{cd}(\mathfrak{a}, R) = 1$  need not have  $\text{ara}(\mathfrak{a}) = 1$ . Hence Theorem 2.2 (ii) genuinely generalizes Theorem 1.4.

**Example 2.3.** Let  $k$  be a field and  $S = k[[X, Y, Z, W]]$ . Consider the elements  $f = XW - YZ$ ,  $g = Y^3 - X^2Z$ , and  $h = Z^3 - Y^2W$  of  $S$ . Let  $R = S/fS$ , and  $\mathfrak{a} = (f, g, h)S/fS$ . Then  $R$  is a noetherian local ring of dimension 3,  $\text{cd}(\mathfrak{a}, R) = 1$ , and  $\text{ara}(\mathfrak{a}) \geq 2$ . See [HeSt, Remark 2.1 (ii)].

### 3. Question 1.3

In this section, we exploit the technique of way-out functors as the main tool to depart from modules to complexes.

**Definition 3.1.** Let  $R$  and  $S$  be two rings, and  $\mathcal{F} : \mathcal{D}(R) \rightarrow \mathcal{D}(S)$  a covariant functor. We say that

- (i)  $\mathcal{F}$  is *way-out left* if for every  $n \in \mathbb{Z}$ , there is an  $m \in \mathbb{Z}$ , such that for any  $R$ -complex  $X$  with  $\sup X \leq m$ , we have  $\sup \mathcal{F}(X) \leq n$ .
- (ii)  $\mathcal{F}$  is *way-out right* if for every  $n \in \mathbb{Z}$ , there is an  $m \in \mathbb{Z}$ , such that for any  $R$ -complex  $X$  with  $\inf X \geq m$ , we have  $\inf \mathcal{F}(X) \geq n$ .
- (iii)  $\mathcal{F}$  is *way-out* if it is both way-out left and way-out right.

The Way-out Lemma appears in [Ha2, Ch. I, Proposition 7.3]. However, we need a refined version which is tailored to our needs. Since the proof of the original result in [Ha2, Ch. I, Proposition 7.3] is left to the reader, we deem it appropriate to include a proof of our refined version for the convenience of the reader as well as bookkeeping.

**Lemma 3.2.** Let  $R$  and  $S$  be two rings, and  $\mathcal{F} : \mathcal{D}(R) \rightarrow \mathcal{D}(S)$  a triangulated covariant functor. Let  $\mathcal{A}$  be an additive subcategory of  $\mathcal{M}(R)$ , and  $\mathcal{B}$  an abelian subcategory of  $\mathcal{M}(S)$  which is closed under extensions. Suppose that  $H_i(\mathcal{F}(M)) \in \mathcal{B}$  for every  $M \in \mathcal{A}$  and every  $i \in \mathbb{Z}$ . Then the following assertions hold:

- (i) If  $X \in \mathcal{D}_{\square}(R)$  with  $H_i(X) \in \mathcal{A}$  for every  $i \in \mathbb{Z}$ , then  $H_i(\mathcal{F}(X)) \in \mathcal{B}$  for every  $i \in \mathbb{Z}$ .
- (ii) If  $\mathcal{F}$  is way-out left and  $X \in \mathcal{D}_{\square}(R)$  with  $H_i(X) \in \mathcal{A}$  for every  $i \in \mathbb{Z}$ , then  $H_i(\mathcal{F}(X)) \in \mathcal{B}$  for every  $i \in \mathbb{Z}$ .
- (iii) If  $\mathcal{F}$  is way-out right and  $X \in \mathcal{D}_{\square}(R)$  with  $H_i(X) \in \mathcal{A}$  for every  $i \in \mathbb{Z}$ , then  $H_i(\mathcal{F}(X)) \in \mathcal{B}$  for every  $i \in \mathbb{Z}$ .
- (iv) If  $\mathcal{F}$  is way-out and  $X \in \mathcal{D}(R)$  with  $H_i(X) \in \mathcal{A}$  for every  $i \in \mathbb{Z}$ , then  $H_i(\mathcal{F}(X)) \in \mathcal{B}$  for every  $i \in \mathbb{Z}$ .

**Proof.** (i): Let  $s = \sup(X)$ . Since  $\text{amp}(X) < \infty$ , we argue by induction on  $n = \text{amp}(X)$ . If  $n = 0$ , then  $X \simeq \Sigma^s H_s(X)$ . Therefore,

$$H_i(\mathcal{F}(X)) \cong H_i(\mathcal{F}(\Sigma^s H_s(X))) \cong H_{i-s}(\mathcal{F}(H_s(X))) \in \mathcal{B},$$

as  $H_s(X) \in \mathcal{A}$ . Now, let  $n \geq 1$  and assume that the result holds for amplitude less than  $n$ . Since  $X \simeq X_{s\subset}$ , there is a distinguished triangle

$$(3.2.1) \quad \Sigma^s H_s(X) \rightarrow X \rightarrow X_{s-1\subset} \rightarrow .$$

It is clear that the two  $R$ -complexes  $\Sigma^s H_s(X)$  and  $X_{s-1\subset}$  have all their homology modules in  $\mathcal{A}$  and their amplitudes are less than  $n$ . Therefore, the induction hypothesis implies that  $H_i(\mathcal{F}(\Sigma^s H_s(X))) \in \mathcal{B}$  and  $H_i(\mathcal{F}(X_{s-1\subset})) \in \mathcal{B}$  for every  $i \in \mathbb{Z}$ . Applying the functor  $\mathcal{F}$  to the distinguished triangle (3.2.1), we get the distinguished triangle

$$\mathcal{F}(\Sigma^s H_s(X)) \rightarrow \mathcal{F}(X) \rightarrow \mathcal{F}(X_{s-1\subset}) \rightarrow ,$$

which in turn yields the long exact homology sequence

$$\begin{aligned} \cdots \rightarrow H_{i+1}(\mathcal{F}(X_{s-1\subset})) \rightarrow H_i(\mathcal{F}(\Sigma^s H_s(X))) \rightarrow H_i(\mathcal{F}(X)) \rightarrow \\ H_i(\mathcal{F}(X_{s-1\subset})) \rightarrow H_{i-1}(\mathcal{F}(\Sigma^s H_s(X))) \rightarrow \cdots . \end{aligned}$$

We break the displayed part of the above exact sequence into the following exact sequences

$$\begin{aligned} H_{i+1}(\mathcal{F}(X_{s-1\subset})) \rightarrow H_i(\mathcal{F}(\Sigma^s H_s(X))) \rightarrow K \rightarrow 0, \\ 0 \rightarrow K \rightarrow H_i(\mathcal{F}(X)) \rightarrow L \rightarrow 0, \\ 0 \rightarrow L \rightarrow H_i(\mathcal{F}(X_{s-1\subset})) \rightarrow H_{i-1}(\mathcal{F}(\Sigma^s H_s(X))) . \end{aligned}$$

Since the subcategory  $\mathcal{B}$  is abelian, we conclude from the first and the third exact sequences above that  $K, L \in \mathcal{B}$ . Since  $\mathcal{B}$  is closed under extensions, the second exact sequence above implies that  $H_i(\mathcal{F}(X)) \in \mathcal{B}$  for every  $i \in \mathbb{Z}$ .

(ii): Let  $i \in \mathbb{Z}$ . Since  $\mathcal{F}$  is way-out left, we can choose an integer  $j \in \mathbb{Z}$  corresponding to  $i - 1$ . Apply the functor  $\mathcal{F}$  to the distinguished triangle

$$X_{\supset j+1} \rightarrow X \rightarrow X_{j\subset} \rightarrow ,$$

to get the distinguished triangle

$$\mathcal{F}(X_{\supset j+1}) \rightarrow \mathcal{F}(X) \rightarrow \mathcal{F}(X_{j\subset}) \rightarrow .$$

From the associated long exact homology sequence, we get

$$0 = H_{i+1}(\mathcal{F}(X_{j\subset})) \rightarrow H_i(\mathcal{F}(X_{\supset j+1})) \rightarrow H_i(\mathcal{F}(X)) \rightarrow H_i(\mathcal{F}(X_{j\subset})) = 0,$$

where the vanishing is due to the choice of  $j$ . Since  $X_{\supset j+1} \in \mathcal{D}_{\square}(R)$  with  $H_i(X_{\supset j+1}) \in \mathcal{A}$  for every  $i \in \mathbb{Z}$ , it follows from (i) that  $H_i(\mathcal{F}(X_{\supset j+1})) \in \mathcal{B}$  for every  $i \in \mathbb{Z}$ , and as a consequence,  $H_i(\mathcal{F}(X)) \in \mathcal{B}$  for every  $i \in \mathbb{Z}$ .

(iii): Given  $i \in \mathbb{Z}$ , choose the integer  $j$  corresponding to  $i + 1$ . The rest of the proof is similar to (ii) using the distinguished triangle

$$X_{\supset j} \rightarrow X \rightarrow X_{j-1\subset} \rightarrow .$$

(iv): Apply the functor  $\mathcal{F}$  to the distinguished triangle

$$X_{\supset 1} \rightarrow X \rightarrow X_{0\subset} \rightarrow ,$$

to get the distinguished triangle

$$\mathcal{F}(X_{\supset 1}) \rightarrow \mathcal{F}(X) \rightarrow \mathcal{F}(X_{0\subset}) \rightarrow .$$

Since  $X_{0\subset} \in \mathcal{D}_{\subset}(R)$  and  $X_{\supset 1} \in \mathcal{D}_{\supset}(R)$  with  $H_i(X_{0\subset}), H_i(X_{\supset 1}) \in \mathcal{A}$  for every  $i \in \mathbb{Z}$ , we deduce from (ii) and (iii) that  $H_i(\mathcal{F}(X_{0\subset})), H_i(\mathcal{F}(X_{\supset 1})) \in \mathcal{B}$  for every  $i \in \mathbb{Z}$ . Using the associated long exact homology sequence, an argument similar to (i) yields that  $H_i(\mathcal{F}(X)) \in \mathcal{B}$  for every  $i \in \mathbb{Z}$ .  $\square$

The next result provides us with a suitable transition device from modules to complexes when dealing with cofiniteness.

**Theorem 3.3.** *If  $\mathfrak{a}$  is an ideal of  $R$ , then the functor  $\mathbf{R}\Gamma_{\mathfrak{a}}(-) : \mathcal{D}(R) \rightarrow \mathcal{D}(R)$  is triangulated and way-out. As a consequence, if  $H_{\mathfrak{a}}^i(M)$  is  $\mathfrak{a}$ -cofinite for every finitely generated  $R$ -module  $M$  and every  $i \geq 0$ , and  $\mathcal{M}(R, \mathfrak{a})_{\text{cof}}$  is an abelian category, then  $H_{\mathfrak{a}}^i(X)$  is  $\mathfrak{a}$ -cofinite for every  $X \in \mathcal{D}^f(R)$  and every  $i \in \mathbb{Z}$ .*

**Proof.** By [Li, Corollary 3.1.4], the functor  $\mathbf{R}\Gamma_{\mathfrak{a}}(-) : \mathcal{D}(R) \rightarrow \mathcal{D}(R)$  is triangulated and way-out. Now, let  $\mathcal{A}$  be the subcategory of finitely generated  $R$ -modules, and let  $\mathcal{B} := \mathcal{M}(R, \mathfrak{a})_{\text{cof}}$ . It can be easily seen that  $\mathcal{B}$  is closed under extensions. It now follows from Lemma 3.2 that  $H_{\mathfrak{a}}^i(X) = H_{-i}(\mathbf{R}\Gamma_{\mathfrak{a}}(X)) \in \mathcal{B}$  for every  $X \in \mathcal{D}^f(R)$  and every  $i \in \mathbb{Z}$ .  $\square$

**Lemma 3.4.** *Suppose that  $R$  admits a dualizing complex  $D$ , and  $\mathfrak{a}$  is an ideal of  $R$ . Further, suppose that  $H_{\mathfrak{a}}^i(Z)$  is  $\mathfrak{a}$ -cofinite for every  $Z \in \mathcal{D}_{\subset}^f(R)$  and every  $i \in \mathbb{Z}$ . Let  $Y \in \mathcal{D}_{\supset}^f(R)$ , and  $X := \mathbf{R}\text{Hom}_R(Y, \mathbf{R}\Gamma_{\mathfrak{a}}(D))$ . Then  $H_i(X)$  is  $\mathfrak{a}$ -cofinite for every  $i \in \mathbb{Z}$ .*

**Proof.** Set  $Z := \mathbf{R}\text{Hom}_R(Y, D)$ . Then clearly,  $Z \in \mathcal{D}_{\subset}^f(R)$ . Let  $\check{C}(\underline{a})$  denote the Čech complex on a sequence of elements  $\underline{a} = a_1, \dots, a_n \in R$  that generates  $\mathfrak{a}$ . For any  $R$ -complex  $W$ , [Li, Proposition 3.1.2] yields that  $\mathbf{R}\Gamma_{\mathfrak{a}}(W) \simeq \check{C}(\underline{a}) \otimes_R^{\mathbf{L}} W$ . Now, by applying the Tensor Evaluation Isomorphism, we get the following display:

$$\begin{aligned} X &= \mathbf{R}\text{Hom}_R(Y, \mathbf{R}\Gamma_{\mathfrak{a}}(D)) \\ &\simeq \check{C}(\underline{a}) \otimes_R^{\mathbf{L}} \mathbf{R}\text{Hom}_R(Y, D) \\ &\simeq \check{C}(\underline{a}) \otimes_R^{\mathbf{L}} Z \\ &\simeq \mathbf{R}\Gamma_{\mathfrak{a}}(Z). \end{aligned}$$

Hence  $H_i(X) \cong H_{\mathfrak{a}}^{-i}(Z)$  for every  $i \in \mathbb{Z}$ , and so the conclusion follows.  $\square$

The next result answers Hartshorne's third question.

**Theorem 3.5.** *Let  $\mathfrak{a}$  be an ideal of  $R$  and  $X \in \mathcal{D}_{\subset}(R)$ . Then the following assertions hold:*

- (i) *If  $H_i(X)$  is  $\mathfrak{a}$ -cofinite for every  $i \in \mathbb{Z}$ , then  $X$  is  $\mathfrak{a}$ -cofinite.*
- (ii) *Assume that  $R$  admits a dualizing complex  $D$ ,  $\mathfrak{a}$  is contained in the Jacobson radical of  $R$ , and  $H_{\mathfrak{a}}^i(Z)$  is  $\mathfrak{a}$ -cofinite for every  $Z \in \mathcal{D}_{\subset}^f(R)$  and every  $i \in \mathbb{Z}$ . If  $X$  is  $\mathfrak{a}$ -cofinite in the sense of Hartshorne, then  $H_i(X)$  is  $\mathfrak{a}$ -cofinite for all  $i \in \mathbb{Z}$ .*

**Proof.** (i) Suppose that  $H_i(X)$  is  $\mathfrak{a}$ -cofinite for all  $i \in \mathbb{Z}$ . The spectral sequence

$$E_{p,q}^2 = \text{Ext}_R^p(R/\mathfrak{a}, H_{-q}(X)) \Rightarrow_p \text{Ext}_R^{p+q}(R/\mathfrak{a}, X)$$

from the proof of [Ha1, Proposition 6.2], together with the assumption that  $E_{p,q}^2$  is finitely generated for every  $p, q \in \mathbb{Z}$ , conspire to imply that  $\text{Ext}_R^{p+q}(R/\mathfrak{a}, X)$  is finitely generated.

On the other hand, one has

$$\mathrm{Supp}_R(X) \subseteq \bigcup_{i \in \mathbb{Z}} \mathrm{Supp}_R(H_i(X)) \subseteq V(\mathfrak{a}).$$

Thus  $X$  is  $\mathfrak{a}$ -cofinite.

(ii) Suppose that  $X$  is  $\mathfrak{a}$ -cofinite in the sense of Hartshorne. Then by definition, there is  $Y \in \mathcal{D}^f(R)$  such that  $X \simeq \mathbf{R}\mathrm{Hom}_R(Y, \mathbf{R}\Gamma_{\mathfrak{a}}(D))$ . Now, the Affine Duality Theorem [Li, Theorem 4.3.1] implies that

$$Y \otimes_R^{\mathbf{L}} \widehat{R}^{\mathfrak{a}} \simeq \mathbf{R}\mathrm{Hom}_R(X, \mathbf{R}\Gamma_{\mathfrak{a}}(D)).$$

Since  $\mathrm{id}_R(\mathbf{R}\Gamma_{\mathfrak{a}}(D)) < \infty$  and  $X \in \mathcal{D}_{\square}(R)$ , we conclude that  $\mathbf{R}\mathrm{Hom}_R(X, \mathbf{R}\Gamma_{\mathfrak{a}}(D)) \in \mathcal{D}_{\square}(R)$ . As the functor  $- \otimes_R \widehat{R}^{\mathfrak{a}} : \mathcal{M}(R) \rightarrow \mathcal{M}(R)$  is faithfully flat, it turns out that  $Y \in \mathcal{D}_{\square}^f(R)$ . Now, the claim follows by Lemma 3.4.  $\square$

**Corollary 3.6.** *Let  $\mathfrak{a}$  be an ideal of  $R$  such that either  $\mathrm{cd}(\mathfrak{a}, R) \leq 1$ , or  $\dim R/\mathfrak{a} \leq 1$ , or  $\dim(R) \leq 2$ . Then the following assertions hold:*

- (i)  $H_{\mathfrak{a}}^i(X)$  is  $\mathfrak{a}$ -cofinite for every  $X \in \mathcal{D}^f(R)$  and every  $i \in \mathbb{Z}$ .
- (ii) Assume that  $R$  admits a dualizing complex  $D$  and  $\mathfrak{a}$  is contained in the Jacobson radical of  $R$ . If  $X \in \mathcal{D}_{\square}(R)$  is  $\mathfrak{a}$ -cofinite in the sense of Hartshorne, then  $H_i(X)$  is  $\mathfrak{a}$ -cofinite for all  $i \in \mathbb{Z}$ .

**Proof.** (i) Follows from Theorem 1.4, [Me2, Corollary 3.14], Theorem 2.2 (ii) and Theorem 3.3.

(ii) Follows by (i) and Theorem 3.5 (ii).  $\square$

#### 4. Correlation between Questions 1.1, 1.2 and 1.3

In this section, we probe the connection between Hartshorne's questions as highlighted in the Introduction.

Some special cases of the following result is more or less proved in [PSY, Theorem 3.10 and Proposition 3.13]. However, we include it here with a different and shorter proof due to its pivotal role in the theory of cofiniteness.

**Lemma 4.1.** *Let  $\mathfrak{a}$  be an ideal of  $R$  and  $X \in \mathcal{D}_{\square}(R)$ . Then the following assertions are equivalent:*

- (i)  $\mathbf{R}\mathrm{Hom}_R(R/\mathfrak{a}, X) \in \mathcal{D}^f(R)$ .
- (ii)  $\mathbf{R}\Gamma_{\mathfrak{a}}(X) \simeq \mathbf{R}\Gamma_{\mathfrak{a}}(Z)$  for some  $Z \in \mathcal{D}_{\square}^f(\widehat{R}^{\mathfrak{a}})$ .
- (iii)  $\mathbf{R}\Gamma_{\mathfrak{a}}(X) \simeq \mathbf{R}\mathrm{Hom}_{\widehat{R}^{\mathfrak{a}}}(Y, \mathbf{R}\Gamma_{\mathfrak{a}}(D))$  for some  $Y \in \mathcal{D}_{\square}^f(\widehat{R}^{\mathfrak{a}})$ , provided that  $\widehat{R}^{\mathfrak{a}}$  enjoys a dualizing complex  $D$ .

**Proof.** (i)  $\Rightarrow$  (ii): By Lemma 2.1,  $Z := \mathbf{L}\Lambda^{\mathfrak{a}}(X) \in \mathcal{D}_{\square}^f(\widehat{R}^{\mathfrak{a}})$ . Then by [AJL, Corollary after (0.3)\*], we have

$$\mathbf{R}\Gamma_{\mathfrak{a}}(Z) \simeq \mathbf{R}\Gamma_{\mathfrak{a}}(\mathbf{L}\Lambda^{\mathfrak{a}}(X)) \simeq \mathbf{R}\Gamma_{\mathfrak{a}}(X).$$

(ii)  $\Rightarrow$  (iii): Set  $Y := \mathbf{R}\mathrm{Hom}_{\widehat{R}^{\mathfrak{a}}}(Z, D)$ . If  $\mathrm{id}_{\widehat{R}^{\mathfrak{a}}}(D) = n$ , then there is a semi-injective resolution  $D \xrightarrow{\sim} I$  of  $D$  such that  $I_i = 0$  for every  $i > \sup D$  or  $i < -n$ . In particular,  $I$

is bounded. On the other hand,  $Z \in \mathcal{D}_{\square}^f(\widehat{R}^{\mathfrak{a}})$ , so there is a bounded  $\widehat{R}^{\mathfrak{a}}$ -complex  $Z'$  such that  $Z \simeq Z'$ . Therefore,

$$Y = \mathbf{R}\mathrm{Hom}_{\widehat{R}^{\mathfrak{a}}}(Z, D) \simeq \mathbf{R}\mathrm{Hom}_{\widehat{R}^{\mathfrak{a}}}(Z', D) \simeq \mathrm{Hom}_{\widehat{R}^{\mathfrak{a}}}(Z', I).$$

But it is obvious that  $\mathrm{Hom}_{\widehat{R}^{\mathfrak{a}}}(Z', I)$  is bounded, so  $Y \in \mathcal{D}_{\square}^f(\widehat{R}^{\mathfrak{a}})$ .

Now, let  $\check{C}(\underline{a})$  denote the Čech complex on a sequence of elements  $\underline{a} = a_1, \dots, a_n \in R$  that generates  $\mathfrak{a}$ . We have

$$\begin{aligned} \mathbf{R}\Gamma_{\mathfrak{a}}(X) &\simeq \mathbf{R}\Gamma_{\mathfrak{a}}(Z) \\ &\simeq \mathbf{R}\Gamma_{\mathfrak{a}}\left(\mathbf{R}\mathrm{Hom}_{\widehat{R}^{\mathfrak{a}}}\left(\mathbf{R}\mathrm{Hom}_{\widehat{R}^{\mathfrak{a}}}(Z, D), D\right)\right) \\ &\simeq \mathbf{R}\Gamma_{\mathfrak{a}}\left(\mathbf{R}\mathrm{Hom}_{\widehat{R}^{\mathfrak{a}}}(Y, D)\right) \\ &\simeq \check{C}(\underline{a}) \otimes_R^{\mathbf{L}} \mathbf{R}\mathrm{Hom}_{\widehat{R}^{\mathfrak{a}}}(Y, D) \\ &\simeq \mathbf{R}\mathrm{Hom}_{\widehat{R}^{\mathfrak{a}}}\left(Y, \check{C}(\underline{a}) \otimes_R^{\mathbf{L}} D\right) \\ &\simeq \mathbf{R}\mathrm{Hom}_{\widehat{R}^{\mathfrak{a}}}(Y, \mathbf{R}\Gamma_{\mathfrak{a}}(D)). \end{aligned}$$

The second isomorphism is due to the fact that  $D$  is a dualizing  $\widehat{R}^{\mathfrak{a}}$ -module, and the fifth isomorphism follows from the application of the Tensor Evaluation Isomorphism. The other isomorphisms are straightforward.

(iii)  $\Rightarrow$  (i): Similar to the argument of the implication (ii)  $\Rightarrow$  (iii), we conclude that  $\mathbf{R}\mathrm{Hom}_{\widehat{R}^{\mathfrak{a}}}(Y, D) \in \mathcal{D}_{\square}^f(\widehat{R}^{\mathfrak{a}})$ . We further have

$$\begin{aligned} \mathbf{L}\Lambda^{\mathfrak{a}}(X) &\simeq \mathbf{L}\Lambda^{\mathfrak{a}}(\mathbf{R}\Gamma_{\mathfrak{a}}(X)) \\ &\simeq \mathbf{L}\Lambda^{\mathfrak{a}}\left(\mathbf{R}\mathrm{Hom}_{\widehat{R}^{\mathfrak{a}}}(Y, \mathbf{R}\Gamma_{\mathfrak{a}}(D))\right) \\ &\simeq \mathbf{L}\Lambda^{\mathfrak{a}}\left(\mathbf{R}\Gamma_{\mathfrak{a}}\left(\mathbf{R}\mathrm{Hom}_{\widehat{R}^{\mathfrak{a}}}(Y, D)\right)\right) \\ &\simeq \mathbf{L}\Lambda^{\mathfrak{a}}\left(\mathbf{R}\mathrm{Hom}_{\widehat{R}^{\mathfrak{a}}}(Y, D)\right) \\ &\simeq \mathbf{R}\mathrm{Hom}_{\widehat{R}^{\mathfrak{a}}}(Y, D) \in \mathcal{D}_{\square}^f(\widehat{R}^{\mathfrak{a}}). \end{aligned}$$

The first and the fourth isomorphisms use [AJL, Corollary after (0.3)\*], the third isomorphism follows from the application of the Tensor Evaluation Isomorphism just as in the previous paragraph, and the fifth isomorphism follows from [PSY, Theorem 1.21], noting that as  $\mathbf{R}\mathrm{Hom}_{\widehat{R}^{\mathfrak{a}}}(Y, D) \in \mathcal{D}_{\square}^f(\widehat{R}^{\mathfrak{a}})$ , its homology modules are  $\mathfrak{a}$ -adically complete  $\widehat{R}^{\mathfrak{a}}$ -modules. Now, the results follows from Lemma 2.1.  $\square$

**Corollary 4.2.** *Let  $\mathfrak{a}$  be an ideal of  $R$  for which  $R$  is  $\mathfrak{a}$ -adically complete and  $X \in \mathcal{D}_{\square}(R)$ . Then the following assertions are equivalent:*

- (i)  $X$  is  $\mathfrak{a}$ -cofinite.
- (ii)  $X \simeq \mathbf{R}\Gamma_{\mathfrak{a}}(Z)$  for some  $Z \in \mathcal{D}_{\square}^f(R)$ .
- (iii)  $X \simeq \mathbf{R}\mathrm{Hom}_R(Y, \mathbf{R}\Gamma_{\mathfrak{a}}(D))$  for some  $Y \in \mathcal{D}_{\square}^f(R)$ , provided that  $R$  enjoys a dualizing complex  $D$ .



**Proof.** For any two  $R$ -complexes  $V \in \mathcal{D}_{\square}^f(R)$  and  $W \in \mathcal{D}_{\square}(R)$ , one may easily see that

$$\mathrm{Supp}_R \left( \mathbf{R} \mathrm{Hom}_R (V, \mathbf{R} \Gamma_{\mathfrak{a}}(W)) \right) \subseteq V(\mathfrak{a}).$$

Also, for any  $U \in \mathcal{D}(R)$ , [Li, Corollary 3.2.1] yields that  $\mathrm{Supp}_R(U) \subseteq V(\mathfrak{a})$  if and only if  $\mathbf{R} \Gamma_{\mathfrak{a}}(U) \simeq U$ . Hence the assertions follow from Lemma 4.1.  $\square$

The next result reveals the correlation between Hartshorne's questions.

**Theorem 4.3.** *Let  $\mathfrak{a}$  be an ideal of  $R$ . Consider the following assertions:*

- (i)  $H_{\mathfrak{a}}^i(M)$  is  $\mathfrak{a}$ -cofinite for every finitely generated  $R$ -module  $M$  and every  $i \geq 0$ , and  $\mathcal{M}(R, \mathfrak{a})_{\mathrm{cof}}$  is an abelian subcategory of  $\mathcal{M}(R)$ .
- (ii)  $H_{\mathfrak{a}}^i(X)$  is  $\mathfrak{a}$ -cofinite for every  $X \in \mathcal{D}^f(R)$  and every  $i \in \mathbb{Z}$ .
- (iii) An  $R$ -complex  $X \in \mathcal{D}_{\square}(R)$  is  $\mathfrak{a}$ -cofinite if and only if  $H_i(X)$  is  $\mathfrak{a}$ -cofinite for every  $i \in \mathbb{Z}$ .

Then the implications (i)  $\Rightarrow$  (ii) and (iii)  $\Rightarrow$  (i) hold. Furthermore, if  $R$  is  $\mathfrak{a}$ -adically complete, then all three assertions are equivalent.

**Proof.** (i)  $\Rightarrow$  (ii): Follows from Theorem 3.3.

(iii)  $\Rightarrow$  (i): Let  $M$  be a finitely generated  $R$ -module. Since  $H_{\mathfrak{a}}^i(M) = 0$  for every  $i < 0$  or  $i > \mathrm{ara}(\mathfrak{a})$ , we have  $\mathbf{R} \Gamma_{\mathfrak{a}}(M) \in \mathcal{D}_{\square}(R)$ . However, [Li, Proposition 3.2.2] implies that

$$\mathbf{R} \mathrm{Hom}_R (R/\mathfrak{a}, \mathbf{R} \Gamma_{\mathfrak{a}}(M)) \simeq \mathbf{R} \mathrm{Hom}_R (R/\mathfrak{a}, M),$$

showing that  $\mathbf{R} \Gamma_{\mathfrak{a}}(M)$  is  $\mathfrak{a}$ -cofinite. The hypothesis now implies that  $H_{\mathfrak{a}}^i(M) = H_{-i}(\mathbf{R} \Gamma_{\mathfrak{a}}(M))$  is  $\mathfrak{a}$ -cofinite for every  $i \geq 0$ .

Now, let  $M$  and  $N$  be two  $\mathfrak{a}$ -cofinite  $R$ -modules and  $f : M \rightarrow N$  an  $R$ -homomorphism. Let  $\varphi : M \rightarrow N$  be the morphism in  $\mathcal{D}(R)$  represented by the roof diagram  $M \xleftarrow{1^M} M \xrightarrow{f} N$ . From the long exact homology sequence associated to the distinguished triangle

$$(4.3.1) \quad M \xrightarrow{\varphi} N \rightarrow \mathrm{Cone}(f) \rightarrow,$$

we deduce that  $\mathrm{Supp}_R(\mathrm{Cone}(f)) \subseteq V(\mathfrak{a})$ . In addition, applying the functor  $\mathbf{R} \mathrm{Hom}_R(R/\mathfrak{a}, -)$  to (4.3.1), gives the distinguished triangle

$$\mathbf{R} \mathrm{Hom}_R(R/\mathfrak{a}, M) \rightarrow \mathbf{R} \mathrm{Hom}_R(R/\mathfrak{a}, N) \rightarrow \mathbf{R} \mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{Cone}(f)) \rightarrow,$$

whose associated long exact homology sequence shows that

$$\mathbf{R} \mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{Cone}(f)) \in \mathcal{D}^f(R).$$

Hence, the  $R$ -complex  $\mathrm{Cone}(f)$  is  $\mathfrak{a}$ -cofinite. However, we have

$$\mathrm{Cone}(f) = \cdots \rightarrow 0 \rightarrow M \xrightarrow{f} N \rightarrow 0 \rightarrow \cdots,$$

so  $\mathrm{Cone}(f) \in \mathcal{D}_{\square}(R)$ . Thus the hypothesis implies that  $H_i(\mathrm{Cone}(f))$  is  $\mathfrak{a}$ -cofinite for every  $i \in \mathbb{Z}$ . It follows that  $\ker f$  and  $\mathrm{coker} f$  are  $\mathfrak{a}$ -cofinite, and as a consequence  $\mathcal{M}(R, \mathfrak{a})_{\mathrm{cof}}$  is an abelian subcategory of  $\mathcal{M}(R)$ .

Now, suppose that  $R$  is  $\mathfrak{a}$ -adically complete.

(ii)  $\Rightarrow$  (iii): Let  $X \in \mathcal{D}_{\square}(R)$ . Suppose that  $H_i(X)$  is  $\mathfrak{a}$ -cofinite for every  $i \in \mathbb{Z}$ . Then Theorem 3.5 (i) yields that  $X$  is  $\mathfrak{a}$ -cofinite.

Conversely, assume that  $X$  is  $\mathfrak{a}$ -cofinite. Then by Corollary 4.2,  $X \simeq \mathbf{R}\Gamma_{\mathfrak{a}}(Z)$  for some  $Z \in \mathcal{D}_{\square}^f(R)$ . Thus the hypothesis implies that

$$H_i(X) \cong H_i(\mathbf{R}\Gamma_{\mathfrak{a}}(Z)) = H_{\mathfrak{a}}^{-i}(Z)$$

is  $\mathfrak{a}$ -cofinite for every  $i \in \mathbb{Z}$ . □

In view of Corollary 4.2, the next result answers Hartshorne's third question for homologically bounded  $R$ -complexes.

**Corollary 4.4.** *Let  $\mathfrak{a}$  be an ideal of  $R$  for which  $R$  is  $\mathfrak{a}$ -adically complete. Suppose that either  $\text{cd}(\mathfrak{a}, R) \leq 1$ , or  $\dim(R/\mathfrak{a}) \leq 1$ , or  $\dim(R) \leq 2$ . Then an  $R$ -complex  $X \in \mathcal{D}_{\square}(R)$  is  $\mathfrak{a}$ -cofinite if and only if  $H_i(X)$  is  $\mathfrak{a}$ -cofinite for every  $i \in \mathbb{Z}$ .*

**Proof.** Obvious in light of Corollary 3.4 and Theorem 4.3. □

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